AD-753 473

0-1 PROGRAMMING FOR COLUMN-CHAINED MATRICES UNDER VECTOR PARTIAL ORDERING

V. J. Bowman

Carnegie-Mellon University

Prepared for:

Office of Naval Research

August 1972

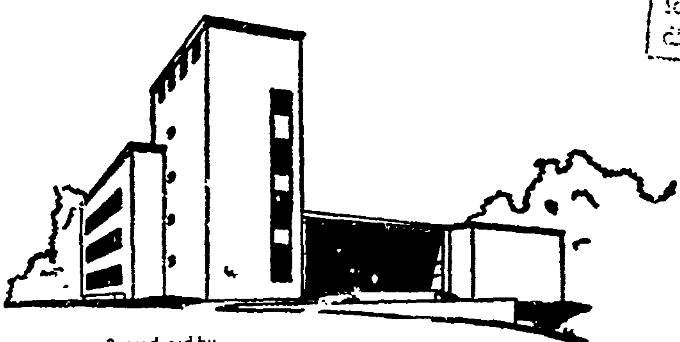
AD758473

Carnegie-Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

WILLIAM LARIMER MELLON, FOUNDER



21

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Graduate School of Industrial Administration Carnegie-Mellon University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP Not Applicable

3. REPORT TITLE

0-1 PROGRAMMING FOR COLUMN-CHAINED MATRICES UNDER VECTOR PARTIAL ORDERING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Management Sciences Research Report July 1972

5. AUTHOR(S) (First name, middle initial, last name)

V.J. Bowman

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1972	15	4

8a. CONTRACT OR GRANT NO. N00014-67-A-0314-0007	9a. ORIGINATOR'S REPORT NUMBER(S) Management Sciences Research Report No.288
b. PROJECT NO. NR 047-048	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) WP-105-71-2
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Br. Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

This paper presents a technique for solving a set of 0-1 programming problems where the columns can be permuted so that all row coefficients are monotone increasing. Such matrices are column-chained under vector partial ordering. The technique is based on equivalence classes that exist on the unit hypercube and for the set of problems described, the approach reduces the set of possible solutions to a subset in which the optimal must lie. An example is presented.

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Column-chained Matrices						
Constraint Equivalence Classes						
0-1 Programming						
Vector Partial Ordering						
Constraint Classification						

DD FORM 1 NOV 65 1473 (BACK)

Unclassified

WP-105-71-2

Management Sciences Research Report No. 288

0-1 PROGRAMMING FOR COLUMN-CHAINED MATRICES UNDER VECTOR PARTIAL ORDERING

by

V.J. Bowman

May 1972
(Revised August 1972)

This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie-Mellon University, under Contract N00014-67-A-0314-0007 NR 047-048 with the U.S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

Management Sciences Research Group
Graduate School of Industrial Administration
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

0-1 PROGRAMMING FOR COLUMN-CHAINED MATRICES
UNDER VECTOR PARTIAL ORDERING

Abstract

This paper presents a technique for solving a set of 0-1 programming problems where the columns can be permuted so that all row coefficients are monotone increasing. Such matrices are column-chained under vector partial ordering. The technique is based on equivalence classes that exist on the unit hypercube and for the set of problems described, the approach reduces the set of possible solutions to a subset in which the optimal must lie. An example is presented.

1. PRELIMINARIES

We wish to consider a special class of the following 0-1 integer programming problem:

$$\begin{aligned} \min \quad & cx \\ \text{st} \quad & Ax \geq b \qquad\qquad (1) \\ & x_i = 0,1, \end{aligned}$$

where A is an mxn matrix.

Definition 1: A matrix A is a column-chained matrix if the columns of A can be linearly ordered under a relationship R.

It should be observed that the property of being a column-chained is dependent on the particular relationship R. Thus, if R is lexicographic ordering, then any matrix A is a column-chained matrix. The ordering relationship that we shall be concerned with is the standard vector partial ordering in R^n, i.e., $a \leqq b$ if $a_i \leq b_i$ for $i = 1,\ldots,n$. If strict inequality must hold in at least one component we write $a \leq b$. The following Lemma exhibits the properties that will be necessary for our discussion.

Lemma 1: A matrix A is a column-chained matrix under vector partial ordering if and only if there exists a permutation $P = (p_1, p_2,\ldots,p_n)$ of the columns of A such that

$$a_{ip_j} \leq a_{ip_{j+1}} \qquad \text{for } j = 1,\ldots,n-1 \text{ and } i = 1,\ldots,n.$$

Proof: This is a direct consequence of a linear ordered set and the ordering relationship.

Column-chained matrices under partial ordering may arise in practical knapsack or capital budgeting problems. For example, by letting $x_i = 1$ if we decide to adopt project i, and 0 if not, the column-chained property reflects the concept that if project i uses more of one resource than project j, project i uses more of all the resources than j. For other techniques for solving general 0-1 knapsack and capital budgeting problems see [2] and [3].

For sake of simplicity, we shall assume that the matrix $\binom{c}{A}$ for our problem (1) is ordered according to Lemma 1, i.e., $P = (1,2,\ldots,n)$.

The solution technique developed herein will depend on equivalence classes for constraints in 0-1 problems. The underlying theory can be found in Bowman [1]. Those results that are of concern to this paper are restated for completeness; however, the proofs are not included.

2. PROPERTIES OF CONSTRAINT EQUIVALENCE CLASSES

1. We say that two constraints $ax \geq b$ and $a'x \geq b'$ are in the same equivalence class E if the set of feasible 0-1 points are the same for $ax \geq b$ and $a'x \geq b'$.
2. There are $(2 \cdot 3^{2^{n-1}} - 2^{2^{n-1}})$ such equivalence classes.
3. The 0-1 vertices can be associated in pairs $\bar{\delta}$ and $-\bar{\delta}$ where $\bar{\delta}_i = \pm 1$. The implication being if $\bar{\delta}_i = +1$ then $x_i = 1$ and if $\bar{\delta}_i = -1$, $x_i = 0$. This transformation is accomplished by $x = 1/2e \pm 1/2\bar{\delta}$ where $e = (1,1,\ldots,1)$. In order to provide uniqueness of identification we require $\Sigma\bar{\delta}_i \geq 0$ and if $\Sigma\bar{\delta}_i = 0$ then $\bar{\delta}_1 = -1$.
4. With every $\bar{\delta}$ pair can be uniquely identified with a vector $p = (p_1,\ldots, p_k)$, $k = [n/2]$ ($[x]$ greatest integer $\leq x$) as follows.

   Let $k^-(\bar{\delta})$ be the number of negative elements of $\bar{\delta}$ then $p(\bar{\delta})$ has the properties:

   a) $0 \leq p_1 \leq p_2 \leq \cdots \leq p_k$

   b) if $p_i \neq 0$ then $p_i < p_{i+1}$

   c) $p_1 = p_2 = \cdots = p_{k-k^-(\bar{\delta})} = 0$

   d) $\delta_i = -1$ if and only if $p_j = i$ for some j.

Thus, the vector $p(\bar{\delta})$ identifies the negative elements of $\bar{\delta}$.

5. If the constraint $ax \geq b$ is such that $0 \leq a_1 \leq a_2 \ldots \leq a_n$ then there exists a function $H(p,v)$ on p and the point $v = \frac{n}{4b'} a$, where $b' = b - (1/2)ae$, such that if $b' > 0$ then

$H(p,v) \geq H(p',v) \quad p \leq p'$ and if $b' \leq 0$ then

$H(p,v) \leq H(p',v) \quad p \leq p'$; that is, H is monotone over the vector partial ordering.

The values of H are calculated as follows:

$$H(p(\bar{\delta}),v) = \begin{cases} 1 \text{ if } \Sigma\bar{\delta}_i v_i > \frac{n}{2} \\ 1 \text{ if } \Sigma\bar{\delta}_i v_i = \frac{n}{2} \text{ and } av \geq b' \\ 0 \text{ if } \Sigma\bar{\delta}_i v_i = \frac{n}{2} \text{ and } av < b' \\ 0 \text{ if } -\frac{n}{2} < \Sigma\bar{\delta}_i v_i < \frac{n}{2} \\ 0 \text{ if } \Sigma\bar{\delta}_i v_i = -\frac{n}{2} \text{ and } av < b' \\ -1 \text{ if } \Sigma\bar{\delta}_i v_i = -\frac{n}{2} \text{ and } av \geq b' \\ -1 \text{ if } \Sigma\bar{\delta}_i v_i < -\frac{n}{2} \end{cases}$$

6. Since $H(p,v)$ is monotone over the partial ordering p, we need know only the following sets to know $H(p,v)$. If $b' \leq 0$

$P^+ = \{p | H(p,v) = 1, \text{ and } H(p',v) = 0, \text{ or } -1 \text{ for } p' \leq p\}$

and $P^- = \{p | H(p,v) = -1 \text{ and } H(p',v) = 0, \text{ or } 1,\ p' \geq p\}$

and if $b' > 0$ then

$P^+ = \{p | H(p,v) = 1 \text{ and } H(p',v) = 0 \text{ or } -1\ p' \geq p\}$

$P^- = \{p | H(p,v) = -1 \text{ and } H(p',v) = 0, \text{ or } +1,\ p' \leq p\}$.

An algorithm for calculating the sets P^+ and P^- is presented in [1].

7. The values of $H(p,v)$ and consequently the sets P^+ and P^- classify $\bar{\delta}$ and $-\bar{\delta}$ and their associated x values into feasible and infeasible sets as follows:

a) $x = 1/2(e + \bar{\delta})$ and $x = 1/2(e - \bar{\delta})$ are <u>both feasible</u> to $ax \geq b$ if and only if $b' \leq 0$ and $H(p(\bar{\delta}),v) = 0$; that is, $b' \leq 0$ and for some $p' \in P^-$ and some $p'' \in P^+$, $p' \leq p(\bar{\delta}) \leq p''$.

b) $x = 1/2(e + \bar{\delta})$ is feasible and $x = 1/2(e - \bar{\delta})$ is infeasible if and only if either

1) $b' > 0$ and $H(p(\bar{\delta}),v) = 1$; that is, $b' > 0$ and for some $p' \in P^+$, $p(\bar{\delta}) \leqq p'$.

or

2) $b' \leq 0$ and $H(p(\bar{\delta}),v) = -1$; that is, $b' \leq 0$ and for some $p' \in P^-$, $p(\bar{\delta}) \leqq p'$.

c) $x = 1/2(e + \bar{\delta})$ is infeasible and $x = 1/2(e - \bar{\delta})$ is feasible if and only if either

1) $b' > 0$ and $H(p(\bar{\delta}),v) = -1$; that is, $b' > 0$ amd for some $p' \in P^-$, $p(\bar{\delta}) \geqq p'$

or

2) $b' \leq 0$ and $H(p(\bar{\delta}), v) = 1$; that is, $b' \leq 0$ and for some $p' \in P^+$, $p(\bar{\delta}) \geqq p'$.

d) $1/2\bar{\delta}$ and $-1/2\bar{\delta}$ are both infeasible if and only if $b' > 0$ and $H(p(\bar{\delta}),v) = 0$; that is, $b' > 0$ and for some $p' \in P^+$ and $p'' \in P^-$, $p' \leq p(\bar{\delta}) \leq p''$.

3. EQUIVALENCE CLASSES ON COLUMN-CHAINED MATRICES

When A is a column-chained matrix under vector partial ordering we note that properties 4 and 5 along with Lemma 1 implies that the functions $H_i(p,v)$ associated with each constraint are monotone for the same vectors p; that is, there is no need to permute the x_i to satisfy $a_1 \leq a_2 \leq \ldots \leq a_n$ for any constraint. Consequently, each constraint associates the points $1/2(e \pm \bar{\delta})$ with exactly the same p. It should be noted that if the matrix is not column-chained (under vector partial ordering) then the points $1/2(e \pm \bar{\delta})$ are not associated with the same vector p for each constraint.

In order to combine the relationships on the vectors p we need to introduce several concepts.

<u>Definition 2</u>: The <u>pomax</u> (<u>pomin</u>) is the operation that chooses the set of maximum (minimum) elements from a partially ordered set.

Of course, since the set is only partially ordered the pomax (pomin) may not be a unique element but, is rather a set of elements. For example, using this notation we can describe the sets P^+ and P^- (from property 6) as

$$P^- = \begin{cases} \text{pomin}\{p \mid H(p,v) = -1\} & \text{if } b' > 0. \\ \text{pomax}\{p \mid H(p,v) = -1\} & \text{if } b' \leq 0. \end{cases}$$

$$P^+ = \begin{cases} \text{pomin}\{p \mid H(p,v) = 1\} & \text{if } b' \leq 0 \\ \text{pomax}\{p \mid H(p,v) = 1\} & \text{if } b' > 0. \end{cases}$$

<u>Definition 3</u>: A <u>descriptive set</u>, of $Ax \geq b$ is a set of sets that completely classify all 0,1 points of $Ax \geq b$ into feasible and infeasible.

For example, if $Ax \geq b$ consists of a single constraint then the sets P^+ and P^- form a descriptive set.

We shall now develop a descriptive set for the constraints of (1) when A is column-chained using equivalence class Property 6. We let $\{P_i^+, P_i^-\}$ be the sets of Property 7 for constraint i, $i = 1,\ldots,n$. We define index sets $I = \{i | b_i' \leq 0\}$ and $J = \{i | b_i' > 0\}$. To facilitate the following development, we define for each constraint $i \in I$

$$P_i^{0+} = \text{pomax}\{p | H(p,v) = 0 \text{ or } -1\}$$

and

$$P_i^{0-} = \text{pomin}\{p | H(p,v) = 0 \text{ or } +1\}$$

These sets are direct by-products of the algorithm to calculate P_i^+ and P_i^- presented in [1] and therefore, involve no additional computations. In order to describe the union of the classifications of a given p, we make the following set definition:

Let $$Q_I^{0+} = \text{pomin}\{p | p \in \bigcup_{i\in I} P_i^{0+}\}$$

$$Q_I^{0-} = \text{pomax}\{p | p \in \bigcup_{i\in I} P_i^{0-}\}$$

$$Q_J^{+} = \text{pomin}\{p | p \in \bigcup_{i\in J} P_i^{+}\} \text{ , and}$$

$$Q_J^{-} = \text{pomax}\{p | p \in \bigcup_{i\in J} P_i^{-}\} \text{ .}$$

These sets then classify the 0.1 points according to the following theorem.

<u>Theorem 1</u>: For the points $1/2(e + \delta)$ and $1/2(e - \delta)$ we have

a) $1/2(e + \delta)$ is feasible if and only if $p(\delta) \leqq p'$ for some $p' \in Q_I^{0+}$ and $p(\delta) \leqq p''$ for some $p'' \in Q_J^+$;

b) $1/2(e - \delta)$ is feasible if and only if $p(\delta) \geqq p'$ for some $p' \in Q_I^{0-}$ and $p(\delta) \geqq p''$ for some $p'' \in Q_J^-$.

<u>Proof</u>: We shall only prove part (a) since the proof for (b) is similar. Assume $1/2(e + \delta)$ is feasible then for all $i \in I$ we have $H_i(p(\delta),v) = -1$ or 0 by Property 7. And, by the monotonicity of $H(\cdot)$ (Property 5) we have

that $H_i(p(\delta),v) = -1$ or 0 for all $i \in I$ if and only if $p(\delta) \leqq p'$ for some $p' \in Q_I^{0+}$. Similarly, for all $i \in J$ we must have $H_i(p(\delta),v) = +1$ and again this is true if and only if $\bar{p}(\delta) \leqq p''$ for some $p'' \in Q_J^+$. Now assume $p(\delta) \leqq p'$ for some $p' \in Q_I^{0+}$ and $p(\delta) \leqq p''$ for some $p'' \in Q_J^+$ by monotonicity $H_i(p(\delta)) = -1$ or 0 for all $i \in I$ and $H_i(p(\delta)) = +1$ for all $i \in J$. By property 7 this implies $1/2(e + \delta)$ is feasible for all constraints. The proof of (b) is similar.

Corollary 1.1: The set of sets $\{Q_I^{0+}, Q_I^{0-}, Q_J^+$ and $Q_J^-\}$ is a descriptive set.

Proof: Obvious.

Corollary 1.2: The points $1/2(e + \delta)$ and $1/2(e - \delta)$ are both feasible if and only if $Q_J^+ = Q_J^- = \emptyset$ (i.e., $J = \emptyset$).

Proof: Assume both points are feasible and Q_J^+ and Q_J^- are not null then $p'' \leq p(\delta) \leq p'$ where $p' \in Q_J^+$ and $p'' \in Q_J^-$; however, by the monotonicity of $H(\cdot)$ we have $p'' \geq p'$ a contradiction.

Having thus been able to classify the feasible and infeasible points of $Ax \geq b$ we now address the problem of optimizing cx over the feasible points.

4. OPTIMIZATION OVER THE DESCRIPTIVE SETS

Problem (1) can [illegible]ted as min cx over the feasible points given by Theorem 1. We assume that the matrix $\binom{c}{A}$ is column-chained under partial ordering. Now since we are interested in minimization, we want a Z^0 such that $\{cx < Z^0, Ax \geq b\}$ has no 0,1 solution while $\{cx \leqq Z^0, Ax \geq b\}$ has at least one solution. This Z^0 is the minimum value. Our approach will be to find the value of Z^0 by examining the equivalence classes of $cx \leq Z$. In order to do this we note that $cx \leq Z$ is equivalent to $-cx \geq -Z$ and substituting $x = e - x'$ we have finally

$$cx' \geq ce - Z. \tag{2}$$

Now since $\binom{c}{A}$ is column-chained we have that $c_1 \leq c_2 \leq \cdots \leq c_n$ so that elements of $p(\delta)$ refer to the same pairs $1/2(e \pm \delta)$ as the constraints. Now let $\{P_Z^+, P_Z^-\}$ be the sets of Property 6 associated with (2). We now need to relate the feasibility information in these sets back to problem 1.

Since $x' = e - x$ and the inferences from the sets $\{P_Z^+, P_Z^-\}$ are on $x' = 1/2(e \pm \bar{\delta})$, we can translate these to the variable x by simple substitution thus,

$$e - 1/2(e \pm \bar{\delta}) = x$$

$$\text{or } x = 1/2(e \mp \bar{\delta}). \tag{3}$$

Equation (3) says that the inferences from property 7 are just the reverse for equation (2). Thus, for example, Property 7b becomes

$x = 1/2(e - \bar{\delta})$ is feasible to $cx \leq Z$ and $x = 1/2(e + \bar{\delta})$ is infeasible to $cx \leq Z$ if and only if either

1) $Z' = 1/2\ ce - Z > 0$ and $H(p(\bar{\delta}),v) = 1$

or

2) $Z' \leq 0$ and $H(p(\bar{\delta}),v) = -1$

As a consequence we have that

$x = 1/2(e - \delta)$ is feasible when either $Z' > 0$ and $H(p,v) = +1$ or $Z' \leq 0$ and $H(p,v) = 0$ or -1 and that

$x = 1/2(e + \delta)$ is feasible when either $Z' > 0$ and $H(p,v) = -1$ or $Z' \leq 0$ and $H(p,v) = 0$ or $+1$.

We can thus, summarize the feasible set for a fixed upper bound Z on the objective function.

Lemma 2: The set of feasible zero-one solutions to $\{cx \leq Z, Ax \geq b\}$ is:

a) $x = 1/2(e + \bar{\delta})$ is feasible if and only if

1) $p(\bar{\delta}) \leqq p''$ for some $p' \in Q_I^{0+}$;

2) $p(\bar{\delta}) \leqq p''$ for $p'' \in Q_J^{+}$;

and 3) (a) if $Z' > 0$, $p(\bar{\delta}) \geqq p^*$ for some $p^* \in P_z^-$

(b) if $Z' \leq 0$, $p(\bar{\delta}) \geq p^*$ for some $p^* \in P_z^-$

b) $x = 1/2(e - \bar{\delta})$ is feasible if and only if

1) $p(\bar{\delta}) \geqq p'$ for some $p' \in Q_I^{0-}$

2) $p(\bar{\delta}) \geqq p''$ for some $p'' \in Q_J^-$

and 3) (a) if $Z' > 0$, $p(\bar{\delta}) \leq p^*$ for some $p^* \in P_z^+$

(b) if $Z' \leq 0$, $p(\bar{\delta}) \leq p^*$ for some $p^* \in P_z^{0+}$.

Proof: Conditions (1) and (2) for each part comes from Theorem 1, Condition 3 is just a restatement of the discussion prior to the lemma.

It is this lemma that is the basis for our analysis of the optimal solution. In order to obtain the optimal value we define the sets

$$Z^+ = \text{pomin}\{p \mid p \in Q_I^{0+} \cup Q_I^+\}$$

$$Z^- = \text{pomax}\{p \mid p \in Q_I^{0-} \cup Q_J^-\}$$

We observe that $x = 1/2(e + \bar{\delta})$ is feasible to $Ax \geq b$ if and only if $p(\bar{\delta}) \leqq p$ for some $p \in Z^+$ and that $x = 1/2(e - \bar{\delta})$ is feasible if and only if $p(\bar{\delta}) \geq p$ for some $p \in Z^-$.

Theorem 2: Let $cx^1 = \min\{cx \mid x = 1/2(e + \bar{\delta})$ and $p(\bar{\delta}) \in Z^+\}$ and $cx^2 = \min\{cx \mid x = 1/2(e - \bar{\delta})$ and $p(\bar{\delta}) \in Z^-\}$ then the optimal solution to (1) is $cx^0 = \min(cx^1, cx^2)$.

Proof: Let $Z^0 = cx^0$ as defined above. Consider the set of feasible solutions to $\{cx \leq Z^0 - \epsilon$ and $Ax \geq b\}$ for $\epsilon > 0$. Now for $p' \in Z^+$, $x = 1/2(e + \bar{\delta})$

is infeasible to $cx \leq Z^0 - \varepsilon$ by definition of Z^0. Consequently, if $(Z^0 - \varepsilon)' > 0$ then $H_Z(p') = +1$ or 0 and for any $p \leq p'$ we have $H_Z(p) = +1$ or 0. Since the only feasible p associated with $x = 1/2(e + \delta)$ are $p \leqq p'$ for some $p' \in Z^+$, there are no points $x = 1/2(e + \delta)$ that are feasible to $Ax \geq b$ and $cx \leq Z^0 - \varepsilon$. On the other hand if $(Z^0 - \varepsilon)' \leq 0$ then $H_Z(p') = -1$ and for any $p \leqq p'$ $H_Z(p) = -1$ again implying that there are no points of the form $x = 1/2(e + \delta)$ that are feasible to both $Ax \geq b$ and $cx \leq Z^0 - \varepsilon$. The proof on Z^- is similar. Thus, there are no solutions to $\{cx \leq Z^0 - \varepsilon, Ax \geq b\}$ for any $\varepsilon > 0$ and there exists one solution, x^0, when $\varepsilon = 0$. Thus, x^0 is the optimum solution, and the theorem is proved.

The implication of Theorem 2 is that by the use of equivalence classes on constraints one can reduce problem (1) (when $\binom{c}{A}$ is column-chained under partial ordering) to the search for an optimal solution over two sets Z^+ and Z^-.

Corollary 2.1: For any c such that $0 \leq c_1 \leq c_2 \leq \ldots \leq c_n$ the optimal lies in the set $Z^+ \cup Z^-$.

This corollary emphasizes the implications of the ordering that as long as we maintain the same constraint set and the cost function changes but preserves the column-chained property of $\binom{c}{A}$ we need only investigate the points associated with $Z^+ \cup Z^-$.

We now illustrate this technique with an example.

Example:

The following knapsack problem is taken from Truath and Woolsey [4].

$$\max \quad 20x_1' + 18x_2' + 17x_3' + 15x_4' + 15x_5' + 10x_6' + 5x_7' + 3x_8' + x_9' + x_{10}'$$
$$30x_1' + 25x_2' + 20x_3' + 18x_4' + 17x_5' + 11x_6' + 5x_7' + 2x_8' + x_9' + x_{10}' \leq 100.$$

In order to make this problem in column-chained format, we take the permutation (10,9,8,7,6,5,4,3,2,1). In addition, we must reverse the sign of the inequality and then substitute $x' = e - x$ to get positive coefficients. We are left with

$$\min \; x_1 + x_2 + 3x_3 + 5x_4 + 10x_5 + 15x_6 + 15x_7 + 17x_8 + 18x_9 + 20x_{10} - 105$$

$$\text{s.t. } x_1 + x_2 + 2x_3 + 5x_4 + 11x_5 + 17x_6 + 18x_7 + 20x_8 + 25x_9 + 30x_{10} \geq +30$$

We note that $b' = b - (1/2)ae = 30 - 65 = -35$ is negative. Using the algorithm in [1] we have

$$P^- = \{(0,0,0,5,7),(0,0,2,5,6),(0,1,2,3,9),(1,2,3,4,8)\},$$

$$P^{0-} = \{(0,0,0,0,10),(0,0,0,4,9),(0,0,0,5,8),(0,0,0,6,7),(0,0,1,5,7),$$
$$(0,0,3,5,6),(0,1,2,5,6)\}$$

$$P^{0+} = \{1,7,8,9,10\}$$

$$P^+ = \emptyset \; .$$

Now points of the form $x = 1/2(e + \delta)$ are feasible for $p \leq (1,7,8,9,10)$ since $P^{0+} = (1,7,8,9,10)$. Note that since n is even (1,7,8,9,10) is the longest P vector and therefore, all $x = 1/2(e + \delta)$ are feasible. Similarly $x = 1/2(e - \delta)$ is feasible for any $p \geq p'$ where $p' \in P^{0-}$. Therefore, we have $Z^+ = P^{0+}$ and $Z^- = P^{0-}$. The table below gives the various values of the objective function for the sets Z^+ and Z^-

Set	P	x's = 1	cx
Z^+	1,7,8,9,10	2,3,4,5,6	34
Z^-	0,0,0,0,10	10	20
	0,0,0,4,9	4,9	23
	0,0,0,5,8	5,8	27
	0,0,0,6,7	6,7	30
	0,0,1,5,7	1,5,7	26
	0,0,3,5,6	3,5,6	28
	0,1,2,5,6	1,2,5,6	27

Examining the values of cx we have $Z^o = 20$ and $x^o = (0,0,0,0,0,0,0,0,0,1)$. Translating back to the original problem, we have the maximizing solution as $x' = (0,1,1,1,1,1,1,1,1,1)$ with a maximum value of 85. We further note that for any c such that $0 \leq c_1 \leq c_2 \leq \cdots \leq c_n$ the optimal solution for the constraint above will be one of the eight points in $Z^+ \cup Z^-$.

5. EXTENSIONS TO REVERSED INEQUALITIES

In the discussion of the optimization procedures we noted that reversing the direction of the inequality had the effect of reversing the feasibility inferences on $+\bar{\delta}$ to $-\bar{\delta}$ and from $-\bar{\delta}$ to $+\bar{\delta}$ under property 7. Consequently, if our problem is of the form

$$\begin{aligned} &\min \quad cx \\ &\text{s.t. } A_1 x \geq b_1 \\ &\qquad A_2 x \leq b_2 \\ &\qquad\ x_i = 0,1 \end{aligned} \tag{4}$$

where $\begin{bmatrix} c \\ A_1 \\ A_2 \end{bmatrix}$ is a column-chained matrix under partial ordering then $A_2 x \leq b_2$ can be treated in the same manner as the objective function in section 4. Consequently, an extension of Lemma 2 provides the set of feasible solutions. First, we define sets

$$Q_{I_1}^{0+} = \text{pomin}\{p \mid p \in \bigcup_{i \in I_1} P_i^{0+}\}$$

$$Q_{I_1}^{0-} = \text{pomax}\{p \mid p \in \bigcup_{i \in I_1} P_i^{0-}\}$$

$$Q_{J_1}^{+} = \text{pomin}\{p \mid p \in \bigcup_{i \in J_1} P_i^{+}\}$$

$$Q_{J_1}^{-} = \text{pomax}\{p \mid p \in \bigcup_{i \in J_1} P_i^{-}\},$$

where the sets $I_1 = \{i \mid b'_{1i} \leq 0\}$, $J_1 = \{i \mid b'_{1i} > 0\}$ and the sets of Q are those defined in section 3 for the constraints $A_1 x \geq b_1$. In a similar manner for constraints $A_2 x \leq b_1$ we define:

$$Q_{I_2}^{0+} = \text{pomin}\{p \mid p \in \bigcup_{i \in I_2} P_i^{0+}\}$$

$$Q^{0-}_{I_2} = \text{pomax}\{p \mid p \in \bigcup_{i \in J_2} P^{0-}_i\}$$

$$Q^{+}_{J_2} = \text{pomin}\{p \mid p \in \bigcup_{i \in J_2} P^{+}_i\} \qquad \text{and}$$

$$Q^{-}_{J_2} = \text{pomax}\{p \mid p \in \bigcup_{i \in J_2} P^{-}_i\}$$

where the sets $I_2 = \{i \mid 1/2a_i e - b_{2i} \leq 0\}$ and $J_2 = \{i \mid 1/2a_i e - b_{2i} > 0\}$.
By arguments similar to that of Section 4 we have

Lemma 3: The set of feasible zero-one solutions to $\{A_1 x \geq b_1$ and $A_2 x \leq b_2\}$ is

a) $x = 1/2(e + \bar{\delta})$ is feasible if and only if

1) $p(\bar{\delta}) \leqq p'$ for some $p' \in Q^{0+}_{I_1}$

2) $p(\bar{\delta}) \leqq p''$ for some $p'' \in Q^{+}_{J_1}$

3) $p(\bar{\delta}) \geqq p^*$ for some $p^* \in Q^{-}_{J_2}$

and 4) $p(\bar{\delta}) \geqq p^{**}$ for some $p^{**} \in Q^{0-}_{I_2}$

b) $x = 1/2(e - \bar{\delta})$ is feasible if and only if

1) $p(\bar{\delta}) \geqq p'$ for some $p' \in Q^{0-}_{I_1}$

2) $p(\bar{\delta}) \geqq p''$ for some $p'' \in Q^{-}_{J_1}$

3) $p(\bar{\delta}) \leqq p^*$ for some $p^* \in Q^{+}_{J_2}$

and 4) $p(\bar{\delta}) \leqq p^{**}$ for some $p^{**} \in Q^{0+}_{I_2}$

We accordingly modify the definitions of Z^+ and Z^- to include the implications of Lemma 3. Thus, we have

$$Z^+ = \text{pomin}\,\{p \mid p \in Q^{0+}_{I_1} \cup Q^{+}_{J_1},\ p \geqq p^*, \text{ and } p \geqq p^{**} \text{ for some}$$
$$p^* \in Q^{-}_{J_2} \text{ and } p^{**} \in Q^{0-}_{I_2}\} \text{ and}$$
$$Z^- = \text{pomax}\,\{p \mid p \in Q^{0-}_{I_1} \cup Q^{-}_{J_1},\ p \leqq p^* \text{ and } p \leqq p^{**} \text{ for some}$$
$$p^* \in Q^{+}_{J_2} \text{ and } p^{**} \in Q^{0+}_{I_2}\}.$$

To optimize the new problem we now apply Theorem 2 to the above sets. Again, the equivalence classes have reduced the solution of (4) to a search over the two sets Z^+ and Z^-.

REFER· 'ES

1. Bowman, V. Joseph, "Constraint Classification on the Unit Hypercube," Working Paper# 92-71-2. Graduate School of Industrial Administration, Carnegie-Mellon University, Management Sciences Research Report No. 287, July 1972.

2. Kolesar, Peter J., "A Branch and Bound Algorithm for the Knapsack Problem," Management Science: Theory 13 No. 9, May 1967, p. 723-735.

3. Nemhauser, G.L. and Z. Ullmann, "Discrete Dynamic Programming and Capital Allocation," Management Science: Theory 15 No. 9, May 1969, p. 494-505.

4. Truath, C.A. and R.E. Woolsey, "Integer Programming: A Study in Computational Efficiency," Management Science: 15 No. 9, May 1969, p. 481-493.